Lecture 36

Computational Electromagnetics, Finite Difference Method, Yee Algorithm

36.1 Introduction to Computational Electromagnetics

Due to the advent of digital computers, numerical methods to seek solutions of Maxwell's equations have become vastly popular. Due to the high fidelity of Maxwell's equations in describing electromagnetic physics in nature, often time, a numerical solution obtained by solving Maxwell's equations are more reliable than laboratory experiments. This field is also known as computational electromagnetics.

Computational electromagnetics consists mainly of two kinds of numerical solvers: one that solves the differential equations directly, the differential-equation solvers; and one that solves the integral equations which are derived from Maxwell's equations.

Differential equation solvers are generally easier to implement, but they solve for the fields directly. The fields permeate all of space, and hence, the unknowns are volumetrically distributed. When the fields are digitized by representing them by their point values in space, they require a large number of unknowns to represent.

On the other hand, the derivation of integral equations require the use of the Green's functions. Green's functions are in general singular when $\mathbf{r} = \mathbf{r}'$, or when the observation point (observation point) $\mathbf{r}$ and the source point $\mathbf{r}'$ coincide. Care has to be taken to discretize the integral equations. However, because the unnknowns are current sources or equivalent sources rather the fields, the unknowns needed are greatly reduced.

There are differences between differential equation solvers and differential equation solvers that will become clearer as we delve into this topic.

36.2 Finite-Difference Method

To obtain the transient (time-domain) solution of the wave equation for a more general, inhomogeneous medium, a numerical method has to be used. The finite-difference time-domain (FDTD) method, a numerical method, is particularly suitable for solving transient problems. Moreover, it is quite versatile, and given the present computer technology, it has been used with great success in solving many practical problems. This method is based on a simple Yee algorithm [194] and has been vastly popularized by Taflove [195, 196].

In the finite-difference method, continuous space-time is replaced with a discrete space-time. Then, in the discrete space-time, partial differential equations are replaced with difference equations. These difference equations are readily implemented on a digital computer. Furthermore, an iterative or time-stepping scheme can be implemented without having to solve large matrices, resulting in a great savings in computer time.[1] More recently, the development of parallel processor architectures in computers has also further enhanced the efficiency of the finite-difference scheme [197].

This method is also described in numerous works (see, for example, Potter 1973 [198]; Taflove 1988 [195]; Ames 2014 [199]; Morton 2019 [200].

36.2.1 The Finite-Difference Approximation

Consider first a scalar wave equation of the form

$$\frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) = \mu(\mathbf{r})\nabla\cdot\mu^{-1}(\mathbf{r})\nabla\phi(\mathbf{r},t). \tag{36.2.1}$$

The above equation appears in scalar acoustic waves in inhomogeneous media [34, 201].

To convert the above into a form that can be solved by a digital computer, one needs to find finite-difference approximations to the time derivatives. Then, the time derivative can be approximated in many ways. For example,

$$\text{Forward difference:}\quad \frac{\partial\phi(\mathbf{r},t)}{\partial t} \approx \frac{\phi(\mathbf{r},t+\Delta t)-\phi(\mathbf{r},t)}{\Delta t}, \tag{36.2.2}$$

$$\text{Backward difference:}\quad \frac{\partial\phi(\mathbf{r},t)}{\partial t} \approx \frac{\phi(\mathbf{r},t)-\phi(\mathbf{r},t-\Delta t)}{\Delta t}, \tag{36.2.3}$$

$$\text{Central difference:}\quad \frac{\partial\phi(\mathbf{r},t)}{\partial t} \approx \frac{\phi(\mathbf{r},t+\frac{\Delta t}{2})-\phi(\mathbf{r},t-\frac{\Delta t}{2})}{\Delta t}, \tag{36.2.4}$$

[1]We shall learn later that most computational electromagnetics methods convert Maxwell's equations into a matrix equation. Then numerical linear algebra methods are used to solve the ensuing matrix equation. Methods where such a matrix is not generated is called a matrix-free method, resulting in great savings in memory.

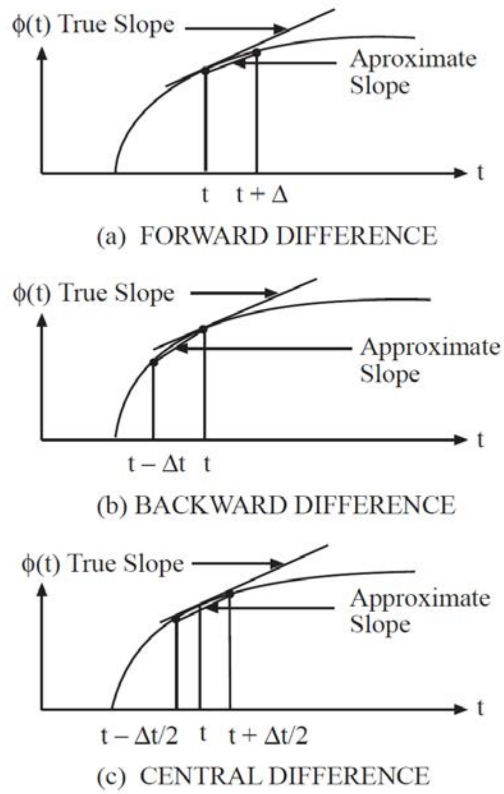

Figure 36.1: Different finite-difference approximations for the time derivative.

where Δt is a small number. Of the three methods of approximating the time derivative, the central-difference scheme is the best approximation, as is evident in Figure 36.1. The errors in the forward and backward differences are $O(\Delta t)$ (first-order error) while the central-difference approximation has an error $O[(\Delta t)^2]$ (second-order error). This can be easily illustrated by Taylor series expanding the right-hand sides of (36.2.2) to (36.2.4).

Consequently, using the central-difference formula twice, we arrive at

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) \approx \frac{\partial}{\partial t}\left[\frac{\phi(\mathbf{r},t+\frac{\Delta t}{2}) - \phi(\mathbf{r},t-\frac{\Delta t}{2})}{\Delta t}\right] \tag{36.2.5}$$

$$\approx \frac{\phi(\mathbf{r},t+\Delta t) - 2\phi(\mathbf{r},t) + \phi(\mathbf{r},t-\Delta t)}{(\Delta t)^2}. \tag{36.2.6}$$

Next, if the function $\phi(\mathbf{r},t)$ is indexed on discrete time steps on the t axis, such that for $t = l\delta t$, then $\phi(\mathbf{r},t) = \phi(\mathbf{r},l\Delta t) = \phi^l(\mathbf{r})$, where l is an integer. Using this notation, Equation

(36.2.6) then becomes

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) \approx \frac{\phi^{l+1}(\mathbf{r}) - 2\phi^l(\mathbf{r}) + \phi^{l-1}(\mathbf{r})}{(\Delta t)^2}. \tag{36.2.7}$$

36.2.2 Time Stepping or Time Marching

With this notation and approximations, Equation (36.2.1) becomes a time-stepping (or time-marching) formula, namely,

$$\phi^{l+1}(\mathbf{r}) = c^2(\mathbf{r})(\Delta t)^2\mu(\mathbf{r})\nabla\cdot\mu^{-1}(\mathbf{r})\nabla\phi^l(\mathbf{r}) + 2\phi^l(\mathbf{r}) - \phi^{l-1}(\mathbf{r}). \tag{36.2.8}$$

Therefore, given the knowledge of $\phi(\mathbf{r},t)$ at $t = l\Delta t$ and $t = (l-1)\Delta t$ for all $\mathbf{r}$, one can deduce $\phi(\mathbf{r},t)$ at $t = (l+1)\Delta t$. In other words, given the initial values of $\phi(\mathbf{r},t)$ at, for example, $t = 0$ and $t = \Delta t$, $\phi(\mathbf{r},t)$ can be deduced for all subsequent times, provided that the time-stepping formula is stable.

At this point, the right-hand side of (36.2.8) involves the space derivatives. There exist a plethora of ways to approximate and calculate the right-hand side of (36.2.8) numerically. Here, we shall illustrate the use of the finite-difference method to calculate the right-hand side of (36.2.8). Before proceeding further, note that the space derivatives on the right-hand side in Cartesian coordinates are

$$\mu(\mathbf{r})\nabla\cdot\mu^{-1}(\mathbf{r})\nabla\phi(\mathbf{r}) = \mu\frac{\partial}{\partial x}\mu^{-1}\frac{\partial}{\partial x}\phi + \mu\frac{\partial}{\partial y}\mu^{-1}\frac{\partial}{\partial y}\phi + \mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi. \tag{36.2.9}$$

Then, one can approximate, using central differencing that

$$\frac{\partial}{\partial z}\phi(x,y,z) \approx \frac{1}{\Delta z}\left[\phi\left(x,y,z+\frac{\Delta z}{2}\right) - \phi\left(x,y,z-\frac{\Delta z}{2}\right)\right], \tag{36.2.10}$$

Consequently, using central differencing two times,

$$\begin{aligned}\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi(x,y,z) \approx \frac{1}{(\Delta z)^2}\bigg\{&\mu^{-1}\left(z+\frac{\Delta z}{2}\right)\phi(x,y,z+\Delta z)\\ &-\left[\mu^{-1}\left(z+\frac{\Delta z}{2}\right)+\mu^{-1}\left(z-\frac{\Delta z}{2}\right)\right]\phi(x,y,z)\\ &+\mu^{-1}\left(z-\frac{\Delta z}{2}\right)\phi(x,y,z-\Delta z)\bigg\}.\end{aligned} \tag{36.2.11}$$

Furthermore, after denoting $\phi(x,y,z) = \phi_{m,n,p}$, $\mu(x,y,z) = \mu_{m,n,p}$, on a discretized grid point at $x = m\Delta x$, $y = n\Delta y$, $z = p\Delta z$, we have $(x,y,z) = (m\Delta x, n\Delta y, p\Delta z)$, and then

$$\begin{aligned}\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi(x,y,z) \approx \frac{1}{(\Delta z)^2}\Big[&\mu^{-1}_{m,n,p+\frac{1}{2}}\phi_{m,n,p+1}\\ &-\left(\mu^{-1}_{m,n,p+\frac{1}{2}}+\mu^{-1}_{m,n,p-\frac{1}{2}}\right)\phi_{m,n,p}+\mu^{-1}_{m,n,p-\frac{1}{2}}\phi_{m,n,p-1}\Big].\end{aligned} \tag{36.2.12}$$

This cumbersome equation can be abbreviated if we define a central difference operator as[2]

$$\bar{\partial}_z \phi_m = \frac{1}{\Delta z}\left(\phi_{m+\frac{1}{2}} - \phi_{m-\frac{1}{2}}\right) \tag{36.2.13}$$

Then the right-hand side of the (36.2.12) can be written succinctly as

$$\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi(x,y,z) \approx \bar{\partial}_z \mu_{m,n,p}\bar{\partial}_z \phi_{m,n,p} \tag{36.2.14}$$

With similar approximations to the other terms in (36.2.9), Equation (36.2.8) becomes

$$\begin{aligned}\phi^{l+1}_{m,n,p} = {}& (\Delta t)^2 c^2_{m,n,p}\mu_{m,n,p}\left[\bar{\partial}_x \mu_{m,n,p}\bar{\partial}_x + \bar{\partial}_y \mu_{m,n,p}\bar{\partial}_y + \bar{\partial}_z \mu_{m,n,p}\bar{\partial}_z\right]\phi_{m,n,p} \\ & + 2\phi^l_{m,n,p} - \phi^{l-1}_{m,n,p}.\end{aligned} \tag{36.2.15}$$

The above can be readily implemented on a computer for time stepping. Notice however, that the use of central differencing results in the evaluation of medium property μ at half grid points. This is inconvenient, as the introduction of half grid points increases computer memory requirements. Hence, it is customary to the medium property at the integer grid points, and to approximate

$$\mu_{m+\frac{1}{2},n,p} \simeq \frac{1}{2}(\mu_{m+1,n,p} + \mu_{m,n,p}), \tag{36.2.16}$$

$$\mu_{m+\frac{1}{2},n,p} + \mu_{m-\frac{1}{2},n,p} \simeq 2\mu_{m,n,p}, \tag{36.2.17}$$

and so on. Moreover, if μ is a smooth function of space, it is easy to show that the errors in the above approximations are of second order by Taylor series expansions.

For a homogeneous medium, with $\Delta x = \Delta y = \Delta z = \Delta s$, (36.2.15) becomes

$$\begin{aligned}\phi^{l+1}_{m,n,p} = {}& \left(\frac{\Delta t}{\Delta s}\right)^2 c^2\left[\phi^l_{m+1,n,p} + \phi^l_{m-1,n,p} + \phi^l_{m,n+1,p} + \phi^l_{m,n-1,p} + \phi^l_{m,n,p+1}\right. \\ & \left. + \phi^l_{m,n,p-1} - 6\phi^l_{m,n,p}\right] + 2\phi^l_{m,n,p} - \phi^{l-1}_{m,n,p}.\end{aligned} \tag{36.2.18}$$

Notice then that with the central-difference approximation, the value of $\phi^{l+1}_{m,n,p}$ is dependent only on $\phi^l_{m,n,p}$, and its nearest neighbors, $\phi^l_{m\pm1,n,p}$, $\phi^l_{m,n\pm1,p}$, $\phi^l_{m,n,p\pm1}$, and $\phi^{l-1}_{m,n,p}$, its value at the previous time step. Moreover, in the finite-difference scheme outlined above, no matrix inversion is required at each time step. Such a scheme is also known as an explicit scheme. The use of an explicit scheme is a major advantage of the finite-difference method compared to the finite-element methods. Consequently, in order to update N grid points using (36.2.15) or (36.2.18), $O(N)$ multiplications are required for each time step. In comparison, $O(N^3)$ multiplications are required to invert an $N \times N$ full matrix, e.g., using Gaussian elimination. The simplicity and efficiency of these algorithms have made them very popular.

[2]This is in the spirit of [202].

36.2.3 Stability Analysis

The implementation of the finite-difference scheme does not always lead to a stable scheme. Hence, in order for the solution to converge, the time-stepping scheme must at least be stable. Consequently, it is useful to find the condition under which a numerical finite-difference scheme is stable. To do this, one performs the von Neumann stability analysis (von Neumann 1943 [203]) on Equation (36.2.18). We will assume the medium to be homogeneous to simplify the analysis.

As shown previously, any wave can be expanded in terms of sum of plane waves in different directions. So if a scheme is not stable for a plane wave, it would not be stable for any wave. Consequently, to perform the stability analysis, we assume a propagating plane wave as a trial solution

$$\phi(x,y,z,t) = A(t)e^{ik_x x + ik_y y + ik_z z}, \tag{36.2.19}$$

In discretized form, it is just

$$\phi^l_{m,n,p} = A^l e^{ik_x m\Delta s + ik_y n\Delta s + ik_z p\Delta s}. \tag{36.2.20}$$

Using (36.2.20), it is easy to show that for the x space derivative,

$$\begin{aligned}\phi^l_{m+1,n,p} - 2\phi^l_{m,n,p} + \phi^l_{m-1,n,p} &= 2[\cos(k_x\Delta s) - 1]\phi^l_{m,n,p} \\ &= -4\sin^2\left(\frac{k_x\Delta s}{2}\right)\phi^l_{m,n,p}.\end{aligned} \tag{36.2.21}$$

The space derivatives in y and z directions can be similarly derived.

The time derivative can be treated and it is proportional to

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t)(\Delta t)^2 \approx \phi^{l+1}_{m,n,p} - 2\phi^l_{m,n,p} + \phi^{l-1}_{m,n,p}. \tag{36.2.22}$$

Substituting (36.2.20) into the above, we have the second time derivative being proportional to

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t)(\Delta t)^2 \approx (A^{l+1} - 2A^l + A^{l-1})e^{ik_x m\Delta s + ik_y n\Delta s + ik_z p\Delta s} \tag{36.2.23}$$

To simplify further, one can assume that

$$A^{l+1} = gA^l. \tag{36.2.24}$$

This is commensurate with assuming that

$$A(t) = A_0 e^{-i\omega t} \tag{36.2.25}$$

where ω can be complex. In other words, our trial solution (36.2.19) is also a time-harmonic signal. If the finite-difference scheme is unstable for such a signal, it is unstable for all signals.

Consequently, the time derivative is proportional to

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t)(\Delta t)^2 \approx (g - 2 + g^{-1})\phi^l_{m,n,p} \tag{36.2.26}$$

We need to find the value of g such that the solution (36.2.20) satisfies (36.2.18). To this end, one uses (36.2.21) and (36.2.24) in (36.2.18), and repeating (36.2.21) for the n and p variables, one obtains

$$\begin{aligned}(g-2+g^{-1})\phi^l_{m,n,p} &= -4\left(\frac{\Delta t}{\Delta s}\right)^2 c^2\left[\sin^2\left(\frac{k_x\Delta s}{2}\right)+\sin^2\left(\frac{k_y\Delta s}{2}\right)\right.\\ &\qquad\left.+\sin^2\left(\frac{k_z\Delta s}{2}\right)\right]\phi^l_{m,n,p}\\ &= -4r^2s^2\phi^l_{m,n,p},\end{aligned} \tag{36.2.27}$$

where

$$r=\left(\frac{\Delta t}{\Delta s}\right)c,\qquad s^2=\sin^2\left(\frac{k_x\Delta s}{2}\right)+\sin^2\left(\frac{k_y\Delta s}{2}\right)+\sin^2\left(\frac{k_z\Delta s}{2}\right). \tag{36.2.28}$$

Equation (16) implies that, for nonzero $\phi^l_{m,n,p}$,

$$g^2-2g+4r^2s^2g+1=0, \tag{36.2.29}$$

or that

$$g=(1-2r^2s^2)\pm 2rs\sqrt{(r^2s^2-1)}\,. \tag{36.2.30}$$

In order for the solution to be stable, it is necessary that $|g|\le 1$. But if

$$r^2s^2<1, \tag{36.2.31}$$

the second term in (36.2.30) is pure imaginary, and

$$|g|^2=(1-2r^2s^2)^2+4r^2s^2(1-r^2s^2)=1, \tag{36.2.32}$$

or stability is ensured. Since $s^2\le 3$ for all k_x, k_y, and k_z, from (36.2.31), one concludes that the general condition for stability is

$$r<\frac{1}{\sqrt{3}},\qquad \text{or}\qquad \Delta t<\frac{\Delta s}{c\sqrt{3}}. \tag{36.2.33}$$

It is clear from the above analysis that for an n-dimensional problem,

$$\Delta t<\frac{\Delta s}{c\sqrt{n}}. \tag{36.2.34}$$

One may ponder on the meaning of this inequality further: but it is only natural that the time step Δt has to be bounded from above. Otherwise, one arrives at the ludicrous notion that the time step can be arbitrarily large thus violating causality. Moreover, if the grid points of the finite-difference scheme are regarded as a simple cubic lattice, then the distance $\Delta s/\sqrt{n}$ is also the distance between the closest lattice planes through the simple cubic lattice. Notice that the time for the wave to travel between these two lattice planes is

$\Delta s/(c\sqrt{n})$. Consequently, the stability criterion (36.2.34) implies that the time step Δt has to be less than the shortest travel time for the wave between the lattice planes in order to satisfy causality. In other words, if the wave is time-stepped ahead of the time on the right-hand side of (36.2.34), instability ensues. The above is also known as the CFL (Courant, Friedrichs, and Lewy 1928 [204]) stability criterion. It could be easily modified for $\Delta x \neq \Delta y \neq \Delta z$.

The above analysis implies that we can pick a larger time step if the space steps are larger. A larger time step will allow one to complete generating a time-domain response rapidly. However, one cannot arbitrary make the space step large due to grid-dispersion error, as shall be discussed next.

36.2.4 Grid-Dispersion Error

When a finite-difference scheme is stable, it still may not produce good results because of the errors in the scheme. Hence, it is useful to ascertain the errors in terms of the size of the grid and the time step. An easy error to analyze is the ***grid-dispersion error***. In a homogeneous, dispersionless medium, all plane waves propagate with the same phase velocity. However, in the finite-difference approximation, all plane waves will not propagate at the same phase velocity due to the grid-dispersion error.

As a consequence, a pulse in the time domain, which is a linear superposition of plane waves with different frequencies, will be distorted if the dispersion introduced by the finite-difference scheme is intolerable. Therefore, to make things simpler, we will analyze the grid-dispersion error in a homogeneous free space medium.

To ascertain the grid-dispersion error, we assume the solution is time-harmonic, or that $A^l = e^{-i\omega l\Delta t}$ in (36.2.20). In this case, the left-hand side of (36.2.27) becomes

$$\left(e^{-i\omega\Delta t} - 2 + e^{+i\omega\Delta t}\right)\phi^l_{m,n,p} = -4\sin^2\left(\frac{\omega\Delta t}{2}\right)\phi^l_{m,n,p}. \tag{36.2.35}$$

Then, from Equation (36.2.27), it follows that

$$\sin\left(\frac{\omega\Delta t}{2}\right) = rs, \tag{36.2.36}$$

where r and s are given in (36.2.28). Now, Equation (36.2.36) governs the relationship between ω and k_x, k_y, and k_z in the finite-difference scheme, and hence, is a dispersion relation.

But if a medium is homogeneous, it is well known that (36.2.1) has a plane-wave solution of the type given by (36.2.19) where

$$\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c|\mathbf{k}| = ck. \tag{36.2.37}$$

where $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$ is the direction of propagation of the plane wave. Defining the phase velocity to be $\omega/k = c$, this phase velocity is isotropic, or the same in all directions. Moreover, it is independent of frequency. But in (36.2.36), because of the definition of s as given by (36.2.28), the dispersion relation between ω and $\mathbf{k}$ is not isotropic. This implies that plane waves propagating in different directions will have different phase velocities.

Equation (36.2.36) departs from Equation (36.2.37) as a consequence of the finite-difference approximation. This departure gives rise to errors, which are the consequence of grid dispersion. For example, when c is a constant, (36.2.37) states that the phase velocities of plane waves of different wavelengths and directions are the same. However, this is not true for (36.2.36), as shall be shown.

Assuming s small, (36.2.36), after using Taylor series expansion, can be written as

$$\frac{\omega\Delta t}{2} = \sin^{-1} rs \cong rs + \frac{r^3 s^3}{6}. \tag{36.2.38}$$

When Δs is small, using the small argument approximation for the sine function, one obtains from (36.2.28)

$$s \simeq \frac{\Delta s}{2}(k_x^2 + k_y^2 + k_z^2)^{1/2} \tag{36.2.39}$$

Equation (36.2.38), by taking the higher-order Taylor expansion of (36.2.38), then becomes

$$\frac{\omega\Delta t}{2} \simeq r\frac{\Delta s}{2}(k_x^2 + k_y^2 + k_z^2)^{1/2}\,[1-\delta] \tag{36.2.40}$$

where (see [34])

$$\delta = \frac{\Delta s^2}{24}\frac{k_x^4 + k_y^4 + k_z^4}{k_x^2 + k_y^2 + k_z^2} + \frac{r^2\Delta s^2}{24}(k_x^2 + k_y^2 + k_z^2) \tag{36.2.41}$$

Since $\mathbf{k}$ is inversely proportional to wavelength λ, then δ in the correction to the above equation is proportional to $\Delta s^2/\lambda^2$. Therefore, to reduce the grid dispersion error, it is necessary to have

$$\left(\frac{\Delta s}{\lambda}\right)^2 \ll 1. \tag{36.2.42}$$

When this is true, using the fact that $r = c\Delta t/\Delta s$, then (36.2.40) becomes

$$\frac{\omega}{c} \approx \sqrt{k_x^2 + k_y^2 + k_z^2}\,. \tag{36.2.43}$$

which is close to the dispersion relation of free space. Consequently, in order for the finite-difference scheme to propagate a certain frequency content accurately, the grid size must be much less than the wavelength of the corresponding frequency. Furthermore, Δt must be chosen so that the CFL stability criterion is met. Hence, the rule of thumb is to choose about 10 to 20 grid points per wavelength. Also, for a plane wave propagating as $e^{i\mathbf{k}\cdot\mathbf{r}}$, an error $\delta\mathbf{k}$ in the vector $\mathbf{k}$ gives rise to cumulative error $e^{i\delta\mathbf{k}\cdot\mathbf{r}}$. The larger the distance traveled, the larger the cumulative phase error, and hence the grid size must be smaller in order to arrest such phase error due to the grid dispersion.

36.3 The Yee Algorithm

The Yee algorithm (Yee 1966 [194]) is specially designed to solve vector electromagnetic field problems on a rectilinear grid. The finite-difference time-domain (FDTD) method (Taflov 1988) when applied to solving electromagnetics problems, usually uses this method. To derive it, Maxwell's equations are first written in Cartesian coordinates:

$$-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \tag{36.3.1}$$

$$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \tag{36.3.2}$$

$$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \tag{36.3.3}$$

$$\frac{\partial D_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_x, \tag{36.3.4}$$

$$\frac{\partial D_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - J_y, \tag{36.3.5}$$

$$\frac{\partial D_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_z. \tag{36.3.6}$$

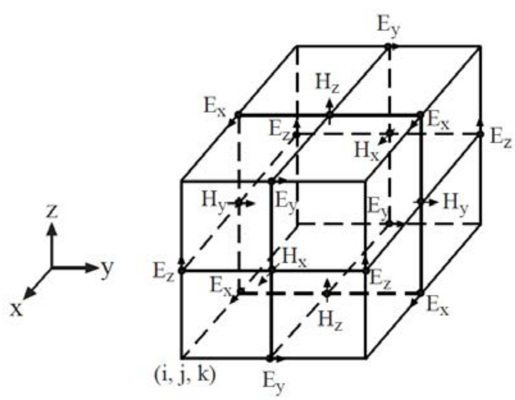

Figure 36.2: The assignment of fields on a grid in the Yee algorithm.

After denoting $f(n\Delta x, m\Delta y, p\Delta z, l\Delta t) = f^l_{m,n,p}$, and replacing derivatives with central finite-differences in accordance with Figure 36.2, (36.3.1) becomes

$$\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}}\right] = \frac{1}{\Delta z}\left[E^l_{y,m,n+\frac{1}{2},p+1} - E^l_{y,m,n+\frac{1}{2},p}\right] - \frac{1}{\Delta y}\left[E^l_{z,m,n+1,p+\frac{1}{2}} - E^l_{z,m,n,p+\frac{1}{2}}\right]. \tag{36.3.7}$$

Moreover, the above can be repeated for (36.3.2) and (36.3.3). Notice that in Figure 36.2, the electric field is always assigned to the edge center of a cube, whereas the magnetic field is always assigned to the face center of a cube.

In fact, after multiplying (36.3.7) by $\Delta z \Delta y$, (36.3.7) is also the approximation of the integral forms of Maxwell's equations when applied at a face of a cube. By doing so, the left-hand side of (36.3.7) becomes

$$(\Delta y \Delta z / \Delta t) \left[B^{l+\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}} \right], \tag{36.3.8}$$

which is the time variation of the total flux through an elemental area $\Delta y \Delta z$. Moreover, by summing this flux on the six faces of the cube shown in Figure 36.2, and using the right-hand side of (36.3.7) and its equivalent, it can be shown that the magnetic flux adds up to zero. Hence, $\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0$ condition is satisfied within the numerical approximations of Yee's algorithm.

Furthermore, a similar approximation of (36.3.4) leads to

$$\begin{aligned} \frac{1}{\Delta t} \left[D^{l}_{x,m+\frac{1}{2},n,p} - D^{l-1}_{x,m+\frac{1}{2},n,p} \right] &= \frac{1}{\Delta y} \left[H^{l-\frac{1}{2}}_{z,m+\frac{1}{2},n+\frac{1}{2},p} - H^{l-\frac{1}{2}}_{z,m+\frac{1}{2},n-\frac{1}{2},p} \right] \\ &\quad - \frac{1}{\Delta z} \left[H^{l-\frac{1}{2}}_{y,m+\frac{1}{2},n,p+\frac{1}{2}} - H^{l-\frac{1}{2}}_{y,m+\frac{1}{2},n,p-\frac{1}{2}} \right] - J^{l-\frac{1}{2}}_{x,m+\frac{1}{2},n,p}. \end{aligned} \tag{36.3.9}$$

Also, similar approximations apply for (36.3.5) and (36.3.6). In addition, the above has an interpretation similar to (36.3.7) if one thinks in terms of a cube that is shifted by half a grid point in each direction. Hence, the approximations of (36.3.4) to (36.3.6) are consistent with the approximation of $\frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{J}$. This way of alternatively solving for the **B** and **D** fields in tandem while the fields are placed on a staggered grid is also called the leap-frog scheme.

In the above, $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. Since the magnetic field and the electric field are assigned on staggered grids, μ and ϵ may have to be assigned on staggered grids. This does not usually lead to serious problems if the grid size is small. Alternatively, (36.2.16) and (36.2.17) can be used to remove this problem.

By eliminating the **E** or the **H** field from the Yee algorithm, it can be shown that the Yee algorithm is equivalent to finite differencing the vector wave equation directly. Hence, the Yee algorithm is also constrained by the CFL stability criterion.

The following figures show some results of FDTD simulations. Because the answers are in the time-domain, beautiful animations of the fields are also available online:

https://www.remcom.com/xfdtd-3d-em-simulation-software

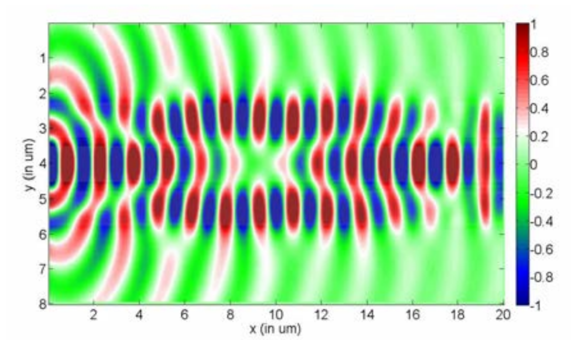

Figure 36.3: The 2D FDTD simulation of complicated optical waveguides (courtesy of Mathworks).

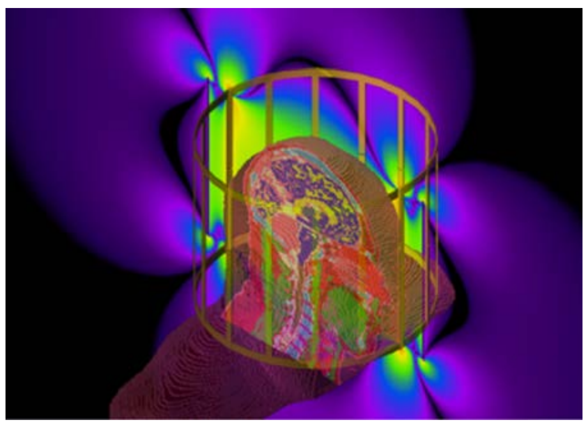

Figure 36.4: FDTD simulation of human head in a squirrel cage of an MRI (magnetic resonance imaging) system (courtesy of REMCOM).

36.3.1 Finite-Difference Frequency Domain Method

Unlike electrical engineering, in many fields, nonlinear problems are prevalent. But when we have a linear time-invariant problem, it is simpler to solve the problem in the frequency domain. This is analogous to perform a time Fourier transform of the pertinent linear equations.

Consequently, one can write (36.3.1) to (36.3.6) in the frequency domain to remove the time derivatives. Then one can apply the finite difference approximation to the space derivatives using the Yee grid. As a result, one arrives at a matrix equation

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b} \tag{36.3.10}$$

where $\mathbf{x}$ is an unknown vector containing $\mathbf{E}$ and $\mathbf{H}$ fields, and $\mathbf{b}$ is a source vector that drives the system containing $\mathbf{J}$. Due to the near-neighbor interactions of the fields on the Yee grid, the matrix $\overline{\mathbf{A}}$ is highly sparse and contains $O(N)$ non-zero elements. When an iterative method is used to solve the above equation, the major cost is in performing a matrix-vector product $\overline{\mathbf{A}} \cdot \mathbf{x}$. However, in practice, the matrix $\overline{\mathbf{A}}$ is never generated nor stored. Because of the simplicity of the Yee algorithm, a code can be written to produce the action of $\overline{\mathbf{A}}$ on $\mathbf{x}$. This can greatly result in memory savings: such methods are called matrix-free methods.

36.4 Absorbing Boundary Conditions

It will not be complete to close this lecture without mentioning absorbing boundary conditions. As computer has finite memory, space of infinitely large extend cannot be simulated with finite computer memory. Hence, it is important to design absorbing boundary conditions at the walls of the simulation domain or box, so that waves impinging on it are not reflected. This mimicks the physics of an infinitely large box.

This is analogous to experments in microwave engineering. In order to perform experiments in an infinite space, such experiments are usually done in an anechoic (non-echoing or non-reflecting) chamber. An anechoic chamber has its walls padded with absorbing materials or microwave absorbers as to minimize the reflections off its walls (see Figure 36.5). Figure 36.6 shows an acoustic version of anechoic chamber.

Figure 36.5: An anechoic chamber for radio frequency. In such an electromagnetically quiet chamber, interference from other RF equipment is minimized (courtesy of Panasonic).

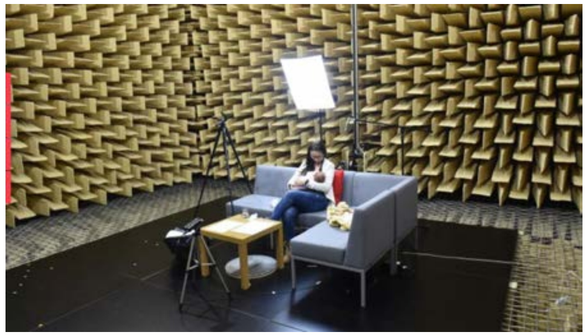

Figure 36.6: An acoustic anechoic chamber. In such a chamber, even the breast-feeding sound of a baby can be heard clearly (courtesy of AGH University, Poland).

By the same token, in order to simulate an infinite box with a finite-size box, absorbing boundary conditions (ABCs) are designed at its walls. The simplest of such ABCs is the impedance boundary condition. (A transmission line terminated with an impedance reflects less than one terminated with an open or a short circuit.) Another simple ABC is to mimick the Sommerfeld radiation condition (much of this is reviewed in [34].

A recently invented ABC is the perfectly matched layers (PML) [205]. Also, another similar ABC is the stretched coordinates PML [206]. Figure 36.7 shows simulation results with and without stretched coordinates PMLs on the walls of the simulation domain [207].

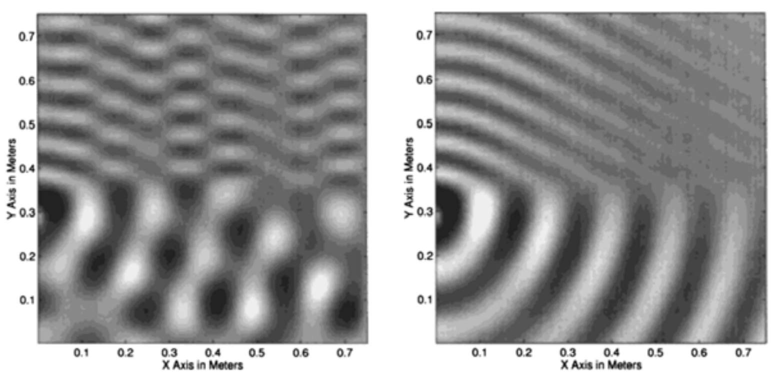

Figure 36.7: Simulation of a source on top of a half-space (left) without stretched coordinates PML; and (right) with stretched coordinats PML [207].

Bibliography

[1] J. A. Kong, *Theory of electromagnetic waves.* New York, Wiley-Interscience, 1975.

[2] A. Einstein *et al.*, "On the electrodynamics of moving bodies," *Annalen der Physik*, vol. 17, no. 891, p. 50, 1905.

[3] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 114, no. 767, pp. 243–265, 1927.

[4] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Physical Review*, vol. 131, no. 6, p. 2766, 1963.

[5] C.-N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," *Physical review*, vol. 96, no. 1, p. 191, 1954.

[6] G. t'Hooft, *50 years of Yang-Mills theory.* World Scientific, 2005.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* Princeton University Press, 2017.

[8] F. Teixeira and W. C. Chew, "Differential forms, metrics, and the reflectionless absorption of electromagnetic waves," *Journal of Electromagnetic Waves and Applications*, vol. 13, no. 5, pp. 665–686, 1999.

[9] W. C. Chew, E. Michielssen, J.-M. Jin, and J. Song, *Fast and efficient algorithms in computational electromagnetics.* Artech House, Inc., 2001.

[10] A. Volta, "On the electricity excited by the mere contact of conducting substances of different kinds. in a letter from Mr. Alexander Volta, FRS Professor of Natural Philosophy in the University of Pavia, to the Rt. Hon. Sir Joseph Banks, Bart. KBPR S," *Philosophical transactions of the Royal Society of London*, no. 90, pp. 403–431, 1800.

[11] A.-M. Ampère, *Exposé méthodique des phénomènes électro-dynamiques, et des lois de ces phénomènes.* Bachelier, 1823.

[12] ——, *Mémoire sur la théorie mathématique des phénomènes électro-dynamiques uniquement déduite de l'expérience: dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et*

26 décembre 1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825. Bachelier, 1825.

[13] B. Jones and M. Faraday, *The life and letters of Faraday.* Cambridge University Press, 2010, vol. 2.

[14] G. Kirchhoff, "Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird," *Annalen der Physik*, vol. 148, no. 12, pp. 497–508, 1847.

[15] L. Weinberg, "Kirchhoff's' third and fourth laws'," *IRE Transactions on Circuit Theory*, vol. 5, no. 1, pp. 8–30, 1958.

[16] T. Standage, *The Victorian Internet: The remarkable story of the telegraph and the nineteenth century's online pioneers.* Phoenix, 1998.

[17] J. C. Maxwell, "A dynamical theory of the electromagnetic field," *Philosophical transactions of the Royal Society of London*, no. 155, pp. 459–512, 1865.

[18] H. Hertz, "On the finite velocity of propagation of electromagnetic actions," *Electric Waves*, vol. 110, 1888.

[19] M. Romer and I. B. Cohen, "Roemer and the first determination of the velocity of light (1676)," *Isis*, vol. 31, no. 2, pp. 327–379, 1940.

[20] A. Arons and M. Peppard, "Einstein's proposal of the photon concept–a translation of the Annalen der Physik paper of 1905," *American Journal of Physics*, vol. 33, no. 5, pp. 367–374, 1965.

[21] A. Pais, "Einstein and the quantum theory," *Reviews of Modern Physics*, vol. 51, no. 4, p. 863, 1979.

[22] M. Planck, "On the law of distribution of energy in the normal spectrum," *Annalen der physik*, vol. 4, no. 553, p. 1, 1901.

[23] Z. Peng, S. De Graaf, J. Tsai, and O. Astafiev, "Tuneable on-demand single-photon source in the microwave range," *Nature communications*, vol. 7, p. 12588, 2016.

[24] B. D. Gates, Q. Xu, M. Stewart, D. Ryan, C. G. Willson, and G. M. Whitesides, "New approaches to nanofabrication: molding, printing, and other techniques," *Chemical reviews*, vol. 105, no. 4, pp. 1171–1196, 2005.

[25] J. S. Bell, "The debate on the significance of his contributions to the foundations of quantum mechanics, Bells Theorem and the Foundations of Modern Physics (A. van der Merwe, F. Selleri, and G. Tarozzi, eds.)," 1992.

[26] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics.* Cambridge University Press, 2018.

[27] C. Pickover, *Archimedes to Hawking: Laws of science and the great minds behind them.* Oxford University Press, 2008.

[28] R. Resnick, J. Walker, and D. Halliday, *Fundamentals of physics.* John Wiley, 1988.

[29] S. Ramo, J. R. Whinnery, and T. Duzer van, *Fields and waves in communication electronics, Third Edition.* John Wiley & Sons, Inc., 1995, also 1965, 1984.

[30] J. L. De Lagrange, "Recherches d'arithmétique," *Nouveaux Mémoires de l'Académie de Berlin*, 1773.

[31] J. A. Kong, *Electromagnetic Wave Theory.* EMW Publishing, 2008, also 1985.

[32] H. M. Schey, *Div, grad, curl, and all that: an informal text on vector calculus.* WW Norton New York, 2005.

[33] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics, Vols. I, II, & III: The new millennium edition.* Basic books, 2011, also 1963, 2006, vol. 1,2,3.

[34] W. C. Chew, *Waves and fields in inhomogeneous media.* IEEE Press, 1995, also 1990.

[35] V. J. Katz, "The history of Stokes' theorem," *Mathematics Magazine*, vol. 52, no. 3, pp. 146–156, 1979.

[36] W. K. Panofsky and M. Phillips, *Classical electricity and magnetism.* Courier Corporation, 2005.

[37] T. Lancaster and S. J. Blundell, *Quantum field theory for the gifted amateur.* OUP Oxford, 2014.

[38] W. C. Chew, "Fields and waves: Lecture notes for ECE 350 at UIUC," *https://engineering.purdue.edu/wcchew/ece350.html*, 1990.

[39] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer Science & Business Media, 2013.

[40] J. M. Crowley, *Fundamentals of applied electrostatics.* Krieger Publishing Company, 1986.

[41] C. Balanis, *Advanced Engineering Electromagnetics.* Hoboken, NJ, USA: Wiley, 2012.

[42] J. D. Jackson, *Classical electrodynamics.* John Wiley & Sons, 1999.

[43] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volumes 1 and 2.* Interscience Publ., 1962.

[44] L. Esaki and R. Tsu, "Superlattice and negative differential conductivity in semiconductors," *IBM Journal of Research and Development*, vol. 14, no. 1, pp. 61–65, 1970.

[45] E. Kudeki and D. C. Munson, *Analog Signals and Systems.* Upper Saddle River, NJ, USA: Pearson Prentice Hall, 2009.

[46] A. V. Oppenheim and R. W. Schafer, *Discrete-time signal processing.* Pearson Education, 2014.

[47] R. F. Harrington, *Time-harmonic electromagnetic fields.* McGraw-Hill, 1961.

[48] E. C. Jordan and K. G. Balmain, *Electromagnetic waves and radiating systems.* Prentice-Hall, 1968.

[49] G. Agarwal, D. Pattanayak, and E. Wolf, "Electromagnetic fields in spatially dispersive media," *Physical Review B*, vol. 10, no. 4, p. 1447, 1974.

[50] S. L. Chuang, *Physics of photonic devices.* John Wiley & Sons, 2012, vol. 80.

[51] B. E. Saleh and M. C. Teich, *Fundamentals of photonics.* John Wiley & Sons, 2019.

[52] M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light.* Elsevier, 2013, also 1959 to 1986.

[53] R. W. Boyd, *Nonlinear optics.* Elsevier, 2003.

[54] Y.-R. Shen, *The principles of nonlinear optics.* New York, Wiley-Interscience, 1984.

[55] N. Bloembergen, *Nonlinear optics.* World Scientific, 1996.

[56] P. C. Krause, O. Wasynczuk, and S. D. Sudhoff, *Analysis of electric machinery.* McGraw-Hill New York, 1986.

[57] A. E. Fitzgerald, C. Kingsley, S. D. Umans, and B. James, *Electric machinery.* McGraw-Hill New York, 2003, vol. 5.

[58] M. A. Brown and R. C. Semelka, *MRI.: Basic Principles and Applications.* John Wiley & Sons, 2011.

[59] C. A. Balanis, *Advanced engineering electromagnetics.* John Wiley & Sons, 1999, also 1989.

[60] Wikipedia, "Lorentz force," https://en.wikipedia.org/wiki/Lorentz_force/, accessed: 2019-09-06.

[61] R. O. Dendy, *Plasma physics: an introductory course.* Cambridge University Press, 1995.

[62] P. Sen and W. C. Chew, "The frequency dependent dielectric and conductivity response of sedimentary rocks," *Journal of microwave power*, vol. 18, no. 1, pp. 95–105, 1983.

[63] D. A. Miller, *Quantum Mechanics for Scientists and Engineers.* Cambridge, UK: Cambridge University Press, 2008.

[64] W. C. Chew, "Quantum mechanics made simple: Lecture notes for ECE 487 at UIUC," *http://wcchew.ece.illinois.edu/chew/course/QMAll20161206.pdf*, 2016.

[65] B. G. Streetman and S. Banerjee, *Solid state electronic devices.* Prentice hall Englewood Cliffs, NJ, 1995.

[66] Smithsonian, "This 1600-year-old goblet shows that the romans were nanotechnology pioneers," https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/, accessed: 2019-09-06.

[67] K. G. Budden, *Radio waves in the ionosphere.* Cambridge University Press, 2009.

[68] R. Fitzpatrick, *Plasma physics: an introduction.* CRC Press, 2014.

[69] G. Strang, *Introduction to linear algebra.* Wellesley-Cambridge Press Wellesley, MA, 1993, vol. 3.

[70] K. C. Yeh and C.-H. Liu, "Radio wave scintillations in the ionosphere," *Proceedings of the IEEE*, vol. 70, no. 4, pp. 324–360, 1982.

[71] J. Kraus, *Electromagnetics.* McGraw-Hill, 1984, also 1953, 1973, 1981.

[72] Wikipedia, "Circular polarization," https://en.wikipedia.org/wiki/Circular_polarization.

[73] Q. Zhan, "Cylindrical vector beams: from mathematical concepts to applications," *Advances in Optics and Photonics*, vol. 1, no. 1, pp. 1–57, 2009.

[74] H. Haus, *Electromagnetic Noise and Quantum Optical Measurements*, ser. Advanced Texts in Physics. Springer Berlin Heidelberg, 2000.

[75] W. C. Chew, "Lectures on theory of microwave and optical waveguides, for ECE 531 at UIUC," *https://engineering.purdue.edu/wcchew/course/tgwAll20160215.pdf*, 2016.

[76] L. Brillouin, *Wave propagation and group velocity.* Academic Press, 1960.

[77] R. Plonsey and R. E. Collin, *Principles and applications of electromagnetic fields.* McGraw-Hill, 1961.

[78] M. N. Sadiku, *Elements of electromagnetics.* Oxford University Press, 2014.

[79] A. Wadhwa, A. L. Dal, and N. Malhotra, "Transmission media," https://www.slideshare.net/abhishekwadhwa786/transmission-media-9416228.

[80] P. H. Smith, "Transmission line calculator," *Electronics*, vol. 12, no. 1, pp. 29–31, 1939.

[81] F. B. Hildebrand, *Advanced calculus for applications.* Prentice-Hall, 1962.

[82] J. Schutt-Aine, "Experiment02-coaxial transmission line measurement using slotted line," http://emlab.uiuc.edu/ece451/ECE451Lab02.pdf.

[83] D. M. Pozar, E. J. K. Knapp, and J. B. Mead, "ECE 584 microwave engineering laboratory notebook," http://www.ecs.umass.edu/ece/ece584/ECE584_lab_manual.pdf, 2004.

[84] R. E. Collin, *Field theory of guided waves.* McGraw-Hill, 1960.

[85] Q. S. Liu, S. Sun, and W. C. Chew, "A potential-based integral equation method for low-frequency electromagnetic problems," *IEEE Transactions on Antennas and Propagation*, vol. 66, no. 3, pp. 1413–1426, 2018.

[86] Wikipedia, "Snell's law," https://en.wikipedia.org/wiki/Snell's_law.

[87] G. Tyras, *Radiation and propagation of electromagnetic waves.* Academic Press, 1969.

[88] L. Brekhovskikh, *Waves in layered media.* Academic Press, 1980.

[89] Scholarpedia, "Goos-hanchen effect," http://www.scholarpedia.org/article/Goos-Hanchen_effect.

[90] K. Kao and G. A. Hockham, "Dielectric-fibre surface waveguides for optical frequencies," in *Proceedings of the Institution of Electrical Engineers*, vol. 113, no. 7. IET, 1966, pp. 1151–1158.

[91] E. Glytsis, "Slab waveguide fundamentals," http://users.ntua.gr/eglytsis/IO/Slab_Waveguides_p.pdf, 2018.

[92] Wikipedia, "Optical fiber," https://en.wikipedia.org/wiki/Optical_fiber.

[93] Atlantic Cable, "1869 indo-european cable," https://atlantic-cable.com/Cables/1869IndoEur/index.htm.

[94] Wikipedia, "Submarine communications cable," https://en.wikipedia.org/wiki/Submarine_communications_cable.

[95] D. Brewster, "On the laws which regulate the polarisation of light by reflexion from transparent bodies," *Philosophical Transactions of the Royal Society of London*, vol. 105, pp. 125–159, 1815.

[96] Wikipedia, "Brewster's angle," https://en.wikipedia.org/wiki/Brewster's_angle.

[97] H. Raether, "Surface plasmons on smooth surfaces," in *Surface plasmons on smooth and rough surfaces and on gratings.* Springer, 1988, pp. 4–39.

[98] E. Kretschmann and H. Raether, "Radiative decay of non radiative surface plasmons excited by light," *Zeitschrift für Naturforschung A*, vol. 23, no. 12, pp. 2135–2136, 1968.

[99] Wikipedia, "Surface plasmon," https://en.wikipedia.org/wiki/Surface_plasmon.

[100] Wikimedia, "Gaussian wave packet," https://commons.wikimedia.org/wiki/File:Gaussian_wave_packet.svg.

[101] Wikipedia, "Charles K. Kao," https://en.wikipedia.org/wiki/Charles_K._Kao.

[102] H. B. Callen and T. A. Welton, "Irreversibility and generalized noise," *Physical Review*, vol. 83, no. 1, p. 34, 1951.

[103] R. Kubo, "The fluctuation-dissipation theorem," *Reports on progress in physics*, vol. 29, no. 1, p. 255, 1966.

[104] C. Lee, S. Lee, and S. Chuang, "Plot of modal field distribution in rectangular and circular waveguides," *IEEE transactions on microwave theory and techniques*, vol. 33, no. 3, pp. 271–274, 1985.

[105] W. C. Chew, *Waves and Fields in Inhomogeneous Media.* IEEE Press, 1996.

[106] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables.* Courier Corporation, 1965, vol. 55.

[107] ——, "Handbook of mathematical functions: with formulas, graphs, and mathematical tables," http://people.math.sfu.ca/~cbm/aands/index.htm.

[108] W. C. Chew, W. Sha, and Q. I. Dai, "Green's dyadic, spectral function, local density of states, and fluctuation dissipation theorem," *arXiv preprint arXiv:1505.01586*, 2015.

[109] Wikipedia, "Very Large Array," https://en.wikipedia.org/wiki/Very_Large_Array.

[110] C. A. Balanis and E. Holzman, "Circular waveguides," *Encyclopedia of RF and Microwave Engineering*, 2005.

[111] M. Al-Hakkak and Y. Lo, "Circular waveguides with anisotropic walls," *Electronics Letters*, vol. 6, no. 24, pp. 786–789, 1970.

[112] Wikipedia, "Horn Antenna," https://en.wikipedia.org/wiki/Horn_antenna.

[113] P. Silvester and P. Benedek, "Microstrip discontinuity capacitances for right-angle bends, t junctions, and crossings," *IEEE Transactions on Microwave Theory and Techniques*, vol. 21, no. 5, pp. 341–346, 1973.

[114] R. Garg and I. Bahl, "Microstrip discontinuities," *International Journal of Electronics Theoretical and Experimental*, vol. 45, no. 1, pp. 81–87, 1978.

[115] P. Smith and E. Turner, "A bistable fabry-perot resonator," *Applied Physics Letters*, vol. 30, no. 6, pp. 280–281, 1977.

[116] A. Yariv, *Optical electronics.* Saunders College Publ., 1991.

[117] Wikipedia, "Klystron," https://en.wikipedia.org/wiki/Klystron.

[118] ——, "Magnetron," https://en.wikipedia.org/wiki/Cavity_magnetron.

[119] ——, "Absorption Wavemeter," https://en.wikipedia.org/wiki/Absorption_wavemeter.

[120] W. C. Chew, M. S. Tong, and B. Hu, "Integral equation methods for electromagnetic and elastic waves," *Synthesis Lectures on Computational Electromagnetics*, vol. 3, no. 1, pp. 1–241, 2008.

[121] A. D. Yaghjian, "Reflections on Maxwell's treatise," *Progress In Electromagnetics Research*, vol. 149, pp. 217–249, 2014.

[122] L. Nagel and D. Pederson, "Simulation program with integrated circuit emphasis," in *Midwest Symposium on Circuit Theory*, 1973.

[123] S. A. Schelkunoff and H. T. Friis, *Antennas: theory and practice.* Wiley New York, 1952, vol. 639.

[124] H. G. Schantz, "A brief history of uwb antennas," *IEEE Aerospace and Electronic Systems Magazine*, vol. 19, no. 4, pp. 22–26, 2004.

[125] E. Kudeki, "Fields and Waves," http://remote2.ece.illinois.edu/~erhan/FieldsWaves/ECE350lectures.html.

[126] Wikipedia, "Antenna Aperture," https://en.wikipedia.org/wiki/Antenna_aperture.

[127] C. A. Balanis, *Antenna theory: analysis and design.* John Wiley & Sons, 2016.

[128] R. W. P. King, G. S. Smith, M. Owens, and T. Wu, "Antennas in matter: Fundamentals, theory, and applications," *NASA STI/Recon Technical Report A*, vol. 81, 1981.

[129] H. Yagi and S. Uda, "Projector of the sharpest beam of electric waves," *Proceedings of the Imperial Academy*, vol. 2, no. 2, pp. 49–52, 1926.

[130] Wikipedia, "Yagi-Uda Antenna," https://en.wikipedia.org/wiki/Yagi-Uda_antenna.

[131] Antenna-theory.com, "Slot Antenna," http://www.antenna-theory.com/antennas/aperture/slot.php.

[132] A. D. Olver and P. J. Clarricoats, *Microwave horns and feeds.* IET, 1994, vol. 39.

[133] B. Thomas, "Design of corrugated conical horns," *IEEE Transactions on Antennas and Propagation*, vol. 26, no. 2, pp. 367–372, 1978.

[134] P. J. B. Clarricoats and A. D. Olver, *Corrugated horns for microwave antennas.* IET, 1984, no. 18.

[135] P. Gibson, "The vivaldi aerial," in *1979 9th European Microwave Conference.* IEEE, 1979, pp. 101–105.

[136] Wikipedia, "Vivaldi Antenna," https://en.wikipedia.org/wiki/Vivaldi_antenna.

[137] ——, "Cassegrain Antenna," https://en.wikipedia.org/wiki/Cassegrain_antenna.

[138] ——, "Cassegrain Reflector," https://en.wikipedia.org/wiki/Cassegrain_reflector.

[139] W. A. Imbriale, S. S. Gao, and L. Boccia, *Space antenna handbook.* John Wiley & Sons, 2012.

[140] J. A. Encinar, "Design of two-layer printed reflectarrays using patches of variable size," *IEEE Transactions on Antennas and Propagation*, vol. 49, no. 10, pp. 1403–1410, 2001.

[141] D.-C. Chang and M.-C. Huang, "Microstrip reflectarray antenna with offset feed," *Electronics Letters*, vol. 28, no. 16, pp. 1489–1491, 1992.

[142] G. Minatti, M. Faenzi, E. Martini, F. Caminita, P. De Vita, D. González-Ovejero, M. Sabbadini, and S. Maci, "Modulated metasurface antennas for space: Synthesis, analysis and realizations," *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 4, pp. 1288–1300, 2014.

[143] X. Gao, X. Han, W.-P. Cao, H. O. Li, H. F. Ma, and T. J. Cui, "Ultrawideband and high-efficiency linear polarization converter based on double v-shaped metasurface," *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 8, pp. 3522–3530, 2015.

[144] D. De Schweinitz and T. L. Frey Jr, "Artificial dielectric lens antenna," Nov. 13 2001, US Patent 6,317,092.

[145] K.-L. Wong, "Planar antennas for wireless communications," *Microwave Journal*, vol. 46, no. 10, pp. 144–145, 2003.

[146] H. Nakano, M. Yamazaki, and J. Yamauchi, "Electromagnetically coupled curl antenna," *Electronics Letters*, vol. 33, no. 12, pp. 1003–1004, 1997.

[147] K. Lee, K. Luk, K.-F. Tong, S. Shum, T. Huynh, and R. Lee, "Experimental and simulation studies of the coaxially fed U-slot rectangular patch antenna," *IEE Proceedings-Microwaves, Antennas and Propagation*, vol. 144, no. 5, pp. 354–358, 1997.

[148] K. Luk, C. Mak, Y. Chow, and K. Lee, "Broadband microstrip patch antenna," *Electronics letters*, vol. 34, no. 15, pp. 1442–1443, 1998.

[149] M. Bolic, D. Simplot-Ryl, and I. Stojmenovic, *RFID systems: research trends and challenges.* John Wiley & Sons, 2010.

[150] D. M. Dobkin, S. M. Weigand, and N. Iyer, "Segmented magnetic antennas for near-field UHF RFID," *Microwave Journal*, vol. 50, no. 6, p. 96, 2007.

[151] Z. N. Chen, X. Qing, and H. L. Chung, "A universal UHF RFID reader antenna," *IEEE transactions on microwave theory and techniques*, vol. 57, no. 5, pp. 1275–1282, 2009.

[152] C.-T. Chen, *Linear system theory and design.* Oxford University Press, Inc., 1998.

[153] S. H. Schot, "Eighty years of Sommerfeld's radiation condition," *Historia mathematica*, vol. 19, no. 4, pp. 385–401, 1992.

[154] A. Ishimaru, *Electromagnetic wave propagation, radiation, and scattering from fundamentals to applications.* Wiley Online Library, 2017, also 1991.

[155] A. E. H. Love, "I. the integration of the equations of propagation of electric waves," *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, vol. 197, no. 287-299, pp. 1–45, 1901.

[156] Wikipedia, "Christiaan Huygens," https://en.wikipedia.org/wiki/Christiaan_Huygens.

[157] ——, "George Green (mathematician)," https://en.wikipedia.org/wiki/George_Green_(mathematician).

[158] C.-T. Tai, *Dyadic Greens Functions in Electromagnetic Theory.* PA: International Textbook, Scranton, 1971.

[159] ——, *Dyadic Green functions in electromagnetic theory.* Institute of Electrical & Electronics Engineers (IEEE), 1994.

[160] W. Franz, "Zur formulierung des huygensschen prinzips," *Zeitschrift für Naturforschung A*, vol. 3, no. 8-11, pp. 500–506, 1948.

[161] J. A. Stratton, *Electromagnetic Theory.* McGraw-Hill Book Company, Inc., 1941.

[162] J. D. Jackson, *Classical Electrodynamics.* John Wiley & Sons, 1962.

[163] W. Meissner and R. Ochsenfeld, "Ein neuer effekt bei eintritt der supraleitfähigkeit," *Naturwissenschaften*, vol. 21, no. 44, pp. 787–788, 1933.

[164] Wikipedia, "Superconductivity," https://en.wikipedia.org/wiki/Superconductivity.

[165] D. Sievenpiper, L. Zhang, R. F. Broas, N. G. Alexopolous, and E. Yablonovitch, "High-impedance electromagnetic surfaces with a forbidden frequency band," *IEEE Transactions on Microwave Theory and techniques*, vol. 47, no. 11, pp. 2059–2074, 1999.

[166] Wikipedia, "Snell's law," https://en.wikipedia.org/wiki/Snell's_law.

[167] H. Lamb, "On sommerfeld's diffraction problem; and on reflection by a parabolic mirror," *Proceedings of the London Mathematical Society*, vol. 2, no. 1, pp. 190–203, 1907.

[168] W. J. Smith, *Modern optical engineering.* McGraw-Hill New York, 1966, vol. 3.

[169] D. C. O'Shea, T. J. Suleski, A. D. Kathman, and D. W. Prather, *Diffractive optics: design, fabrication, and test.* Spie Press Bellingham, WA, 2004, vol. 62.

[170] J. B. Keller and H. B. Keller, "Determination of reflected and transmitted fields by geometrical optics," *JOSA*, vol. 40, no. 1, pp. 48–52, 1950.

[171] G. A. Deschamps, "Ray techniques in electromagnetics," *Proceedings of the IEEE*, vol. 60, no. 9, pp. 1022–1035, 1972.

[172] R. G. Kouyoumjian and P. H. Pathak, "A uniform geometrical theory of diffraction for an edge in a perfectly conducting surface," *Proceedings of the IEEE*, vol. 62, no. 11, pp. 1448–1461, 1974.

[173] R. Kouyoumjian, "The geometrical theory of diffraction and its application," in *Numerical and Asymptotic Techniques in Electromagnetics.* Springer, 1975, pp. 165–215.

[174] S.-W. Lee and G. Deschamps, "A uniform asymptotic theory of electromagnetic diffraction by a curved wedge," *IEEE Transactions on Antennas and Propagation*, vol. 24, no. 1, pp. 25–34, 1976.

[175] Wikipedia, "Fermat's principle," https://en.wikipedia.org/wiki/Fermat's_principle.

[176] N. Yu, P. Genevet, M. A. Kats, F. Aieta, J.-P. Tetienne, F. Capasso, and Z. Gaburro, "Light propagation with phase discontinuities: generalized laws of reflection and refraction," *Science*, vol. 334, no. 6054, pp. 333–337, 2011.

[177] A. Sommerfeld, *Partial differential equations in physics.* Academic Press, 1949, vol. 1.

[178] R. Haberman, *Elementary applied partial differential equations.* Prentice Hall Englewood Cliffs, NJ, 1983, vol. 987.

[179] G. A. Deschamps, "Gaussian beam as a bundle of complex rays," *Electronics letters*, vol. 7, no. 23, pp. 684–685, 1971.

[180] J. Enderlein and F. Pampaloni, "Unified operator approach for deriving hermite–gaussian and laguerre–gaussian laser modes," *JOSA A*, vol. 21, no. 8, pp. 1553–1558, 2004.

[181] D. L. Andrews, *Structured light and its applications: An introduction to phase-structured beams and nanoscale optical forces.* Academic Press, 2011.

[182] J. W. Strutt, "Xv. on the light from the sky, its polarization and colour," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 41, no. 271, pp. 107–120, 1871.

[183] L. Rayleigh, "X. on the electromagnetic theory of light," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 12, no. 73, pp. 81–101, 1881.

[184] R. C. Wittmann, "Spherical wave operators and the translation formulas," *IEEE Transactions on Antennas and Propagation*, vol. 36, no. 8, pp. 1078–1087, 1988.

[185] S. Sun, Y. G. Liu, W. C. Chew, and Z. Ma, "Calderón multiplicative preconditioned efie with perturbation method," *IEEE Transactions on Antennas and Propagation*, vol. 61, no. 1, pp. 247–255, 2012.

[186] G. Mie, "Beiträge zur optik trüber medien, speziell kolloidaler metallösungen," *Annalen der physik*, vol. 330, no. 3, pp. 377–445, 1908.

[187] Wikipedia, "Mie scattering," https://en.wikipedia.org/wiki/Mie_scattering.

[188] R. E. Collin, *Foundations for microwave engineering.* John Wiley & Sons, 2007, also 1966.

[189] L. B. Felsen and N. Marcuvitz, *Radiation and scattering of waves.* John Wiley & Sons, 1994, also 1973, vol. 31.

[190] P. P. Ewald, "Die berechnung optischer und elektrostatischer gitterpotentiale," *Annalen der physik*, vol. 369, no. 3, pp. 253–287, 1921.

[191] E. Whitaker and G. Watson, *A Course of Modern Analysis.* Cambridge Mathematical Library, 1927.

[192] A. Sommerfeld, *Über die Ausbreitung der Wellen in der drahtlosen Telegraphie.* Verlag der Königlich Bayerischen Akademie der Wissenschaften, 1909.

[193] J. Kong, "Electromagnetic fields due to dipole antennas over stratified anisotropic media," *Geophysics*, vol. 37, no. 6, pp. 985–996, 1972.

[194] K. Yee, "Numerical solution of initial boundary value problems involving maxwell's equations in isotropic media," *IEEE Transactions on Antennas and Propagation*, vol. 14, no. 3, pp. 302–307, 1966.

[195] A. Taflove, "Review of the formulation and applications of the finite-difference time-domain method for numerical modeling of electromagnetic wave interactions with arbitrary structures," *Wave Motion*, vol. 10, no. 6, pp. 547–582, 1988.

[196] A. Taflove and S. C. Hagness, *Computational electrodynamics: the finite-difference time-domain method.* Artech house, 2005, also 1995.

[197] W. Yu, R. Mittra, T. Su, Y. Liu, and X. Yang, *Parallel finite-difference time-domain method.* Artech House Norwood, 2006.

[198] D. Potter, "Computational physics," 1973.

[199] W. F. Ames, *Numerical methods for partial differential equations.* Academic press, 2014, also 1977.

[200] K. W. Morton, *Revival: Numerical Solution Of Convection-Diffusion Problems (1996).* CRC Press, 2019.

[201] K. Aki and P. G. Richards, *Quantitative seismology*, 2002.

[202] W. C. Chew, "Electromagnetic theory on a lattice," *Journal of Applied Physics*, vol. 75, no. 10, pp. 4843–4850, 1994.

[203] J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik, Berlın.* Springer, New York, Dover Publications, 1943.

[204] R. Courant, K. Friedrichs, and H. Lewy, "Über die partiellen differenzengleichungen der mathematischen physik," *Mathematische annalen*, vol. 100, no. 1, pp. 32–74, 1928.

[205] J.-P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *Journal of computational physics*, vol. 114, no. 2, pp. 185–200, 1994.

[206] W. C. Chew and W. H. Weedon, "A 3d perfectly matched medium from modified maxwell's equations with stretched coordinates," *Microwave and optical technology letters*, vol. 7, no. 13, pp. 599–604, 1994.

[207] W. C. Chew, J. Jin, and E. Michielssen, "Complex coordinate system as a generalized absorbing boundary condition," in *IEEE Antennas and Propagation Society International Symposium 1997. Digest*, vol. 3. IEEE, 1997, pp. 2060–2063.

[208] W. C. H. McLean, *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, 2000.

[209] G. C. Hsiao and W. L. Wendland, *Boundary integral equations.* Springer, 2008.

[210] K. F. Warnick, *Numerical analysis for electromagnetic integral equations.* Artech House, 2008.

[211] M. M. Botha, "Solving the volume integral equations of electromagnetic scattering," *Journal of Computational Physics*, vol. 218, no. 1, pp. 141–158, 2006.

[212] P. K. Banerjee and R. Butterfield, *Boundary element methods in engineering science.* McGraw-Hill London, 1981, vol. 17.

[213] O. C. Zienkiewicz, R. L. Taylor, P. Nithiarasu, and J. Zhu, *The finite element method.* McGraw-Hill London, 1977, vol. 3.

[214] J.-F. Lee, R. Lee, and A. Cangellaris, "Time-domain finite-element methods," *IEEE Transactions on Antennas and Propagation*, vol. 45, no. 3, pp. 430–442, 1997.

[215] J. L. Volakis, A. Chatterjee, and L. C. Kempel, *Finite element method electromagnetics: antennas, microwave circuits, and scattering applications.* John Wiley & Sons, 1998, vol. 6.

[216] J.-M. Jin, *The finite element method in electromagnetics.* John Wiley & Sons, 2015.

[217] G. Strang, *Linear algebra and its applications.* Academic Press, 1976.

[218] Cramer and Gabriel, *Introduction a l'analyse des lignes courbes algebriques par Gabriel Cramer...* chez les freres Cramer & Cl. Philibert, 1750.

[219] J. A. Schouten, *Tensor analysis for physicists.* Courier Corporation, 1989.

[220] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical recipes 3rd edition: The art of scientific computing.* Cambridge University Press, 2007.